Math 6000, Fall 2020 (Prof. Kinser), Homework 2

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Source Discussed solutions with Zach after thinking about the hw myself and then did proof analysis of each other's paper after we had written our own initial drafts.

Problem 1. *Skills developed: Interpreting and testing an abstract definition in familiar settings.*

In each category below, decide whether there exists a free object on arbitrary set X. If so, prove it by constructing the free object and demonstrating the definition holds. If not, choose a specific set X and prove that no free object on X can exist.

Each category below is a familiar concrete category. So just treat the objects as having underlying sets as you usually would, without writing U for the "underlying set" functor

(a) The category **Sets** of all sets.

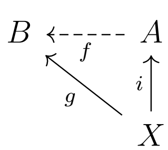
(b) The category **Fields** of all fields.

(c) The category **Rings** of comm. rings with $1 (\neq 0)$ and homomorphisms which preserve 1.

(d) The category **Top** of topological spaces and continuous functions.

Defs/Theorems 1. Given a set X, object $A \in \mathcal{C}$ and morphisms of sets $i : X \to A$. We say that A is a **free object** on X (and X is a basis of A) if it satisfies the following **universal property:**

1a. Given any map of sets $g : X \to B$ where $B \in \mathcal{C}$, there exists unique morphism $f : A \to B$ such that diagram commutes (i.e. $\exists! f$ such that $g = f \circ i$)



2. The discrete topology is the finest topology that can be given on a set, i.e., it defines all the subsets as open sets. In particular, each singleton is an open set in the discrete topology.

3. A field homomorphism $\phi : F \to F'$ is identically 0 or injective.

3b. If there is a homomorphism between two fields, then they have the same characteristic.

(a) Yes, there does exist a free object on arbitrary set X in the category of **Sets.**

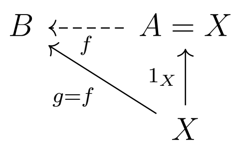
Since every object in a category has an identity morphism, $X \in$ Ob(**Sets**) has an identity morphism, namely $1_X \in Hom_{\textbf{Sets}}(X, X)$.

Let A be an object in **Sets.** Consider $A = X$. Then, $i : X \to A$ is precisely the identity morphism $1_X : X \to X$.

Hence, given $g : X \to B \in$ **Sets**, there exists a unique morphism $f : X \to B$ such that $g = f \circ 1_X$ (since the identity is unique, f has to be unique).

$\therefore$, in **Sets**, $A = X$ is the free object on X and $f = g$ is the unique morphism from A to B.

Here is the corresponding commutative diagram:



(b) **Claim:** Free object on $X \neq \emptyset$ does not exist in a category of **Fields**.

By way of contradiction, suppose we have $i : X \to A$ with A free on X. Let A be a field of characteristic p.

Let $g : X \to B$, be the given morphism, where B is a field of characteristic $q \neq p$.

Then, there should be a unique morphism $f : A \to B$ such that $g = f \circ i$. This is precisely the contradiction since f is a homomorphism between fields, and A and B have different characteristic.

(Note, if the category is a field of a fixed characteristic, there may be a free object if the arbitrary set $X = \emptyset$, but this is not true generally).

(c) Yes, there does exist a free object on arbitrary set X in the category of **Rings**.

Consider the polynomial ring $\mathbb{Z}[x]$. By definition, it is the set of all formal sums $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ with $n \geq 0$ and each $a_i \in \mathbb{Z}$.

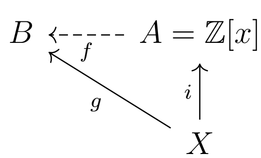
Let X be any arbitrary set and let $i : X \to A$. Given a $g : X \to B$, we want to show there is a unique morphism $f : A \to B$.

Since A and B are both rings and we assume that homomorphism preserves $1, f(1) = 1$. Hence, $g = f \circ i$ exists.

Now, suppose there is another such ring homomorphsim $\hat{f} : A \to B$. Then, we have $f \circ i = g = \hat{f} \circ i \Rightarrow f \circ i = \hat{f} \circ i \Rightarrow f = \hat{f}$. Hence, f is a unique morphism.

(Credit: The motivation for this problem came entirely from discussion with Zach).

Here is the corresponding commutative diagram:



(d) Yes, there does exist a free object on arbitrary set X in the category of **Top**.

Suppose $A = (X, \tau)$ is endowed with the discrete topology. Let $(B, \tau_2) \in Ob(\textbf{Top})$.

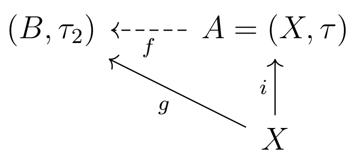
We assert that A is a free object. Suppose $V \subset B$ is an open set. Consider $f^{-1}(V) \subset A$. Since every subset of A is open, $f^{-1}(V) \subset A$ is open.

Since the preimage of an open set is open, f is continuous. In particular, $g = f \circ i$.

Now, suppose there exists another map $\overline{f}$ such that $g = \overline{f} \circ i$.

Then $f \circ i = \overline{f} \circ i \Rightarrow f = \overline{f}$.

Here is the corresponding commutative diagram:



Problem 2. *Skills developed: Construction of a categorical equivalence, and practice with matrix rings and modules.*

Let K be a field, and $K-$Mod the category of $K-$modules (i.e. vector spaces). Let $R = Mat_{2\times 2}(K)$ be the ring of 2×2 matrices over K, and $R-$Mod the category of left $R-$modules. We will show that $K-$Mod and $R-$Mod are equivalent categories, despite that fact that K and R are clearly not isomorphic rings.

(a) Define a map on objects $F : K - \text{Mod} \to R - \text{Mod}$ by sending a vector space V to the $R-$modules $V \oplus V$, where R acts on $(v_1, v_2) \in V \oplus V$ by the standard matrix multiplication formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

Show how to make F a functor in the most natural way.

(b) Let e be the primitive idempotent in R, for concreteness let's take $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Check (but don't turn in) that the ring eRe is isomorphic to the field K, where $a \in K$ is identified with the matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Also check that eM is a left $eRe-$module, and thus can be considered as a K- vector space. Therefore, we can define a map on objects $G : R - \text{Mod} \to K - \text{Mod}$ by sending an $R-$module M to eM. Show how to make G a functor in the most natural way.

(c) It is easy to see that GF isomorphic to the identity functor on $K - \text{Mod}$. (Check this but don't turn it in.) On the other hand, FG is not exactly the identity functor, but $FG(M) \simeq M$ for all $M \in R - \text{Mod}$ Show that the functor FG is isomorphic to the identity functor on $R-$mod. This shows that $R - \text{mod}$ and $K - \text{mod}$ are equivalent categories.

This generalizes to $n \times n$ matrices over arbitrary rings with essentially the same proof. In general, two rings S_1, S_2 such that the categories S_1-mod and S_2-mod are equivalent are said to be "Morita equivalent" rings.

Defs/Theorems 1. Let R be a ring (not necessarily commutative nor with 1). A *left R-module* or a *left-module* over R is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies:

(a) $(r + s)m = rm + sm$, for all $r, s \in R, m \in M$.

(b) $(rs)m = r(sm)$, for all $r, s \in R, m \in M$, and

(c) $r(m + n) = rm + rn$, for all $r \in R, m, n \in M$.

If the ring R has a 1 we impose the additional axiom:

(d) $1m = m$, for all $m \in M$.

2. Suppose V and W are vector spaces over the field K. The cartesian product $V \times W$ can be given the structure of a vector space over K by defining the operations componentwise:

(i) $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$

(ii) $\alpha(v, w) = (\alpha v, \alpha w)$.

for $v, v_1, v_2 \in V, w, w_1, w_2 \in W$, and $\alpha \in K$.

The resulting vector space is called the **direct sum** of V and W and is usually denoted by $V \bigoplus W$.

3. (Pg. 327, Prop 10.2.2 D and F) Let M, N, and L be $R-$modules.

(1) A map $\phi : M \to N$ is an $R-$module homomorphism iff $\phi(rx + y) = r\phi(x) + \phi(y)$ for all $x, y \in M$ and all $r \in R$.

(2) Let ϕ, ψ be elements of $Hom_R(M, n)$. Define $\phi + \psi$ by

$$(\phi + \psi)(m) = \phi(m) + \psi(m) \text{ for all } m \in M$$

Then $\phi + \psi \in Hom_R(M, N)$ and with this operation $Hom_R(M, n)$ is an abelian group. If R is a commutative ring then for $r \in R$ define $r\phi$ by

$$(r\phi)(m) = r(\phi(m)) \quad \text{for all } m \in M$$

.

Proof (a)

Show Show that F is a functor, i.e.

(1) $\forall\, V \in Ob(K - \text{Mod}), F(R) \in Ob(R - \text{Mod})$

(2) $\forall\ \phi \in Hom_{K-\text{Mod}}(V_1, V_2), F(\phi) \in Hom_{R-\text{Mod}}(F(V_1), F(V_2))$ such that $F(1_{V_1}) = 1_{F(V_1)}$.

(3) $F(\phi_2 \circ \phi_1) = F(\phi_2) \circ F(\phi_1)$ for all ϕ_2, ϕ_1 composable in $K - \text{Mod}$.

(a-i) Let $V \in Ob(K - \text{Mod})$. Define $F(V) = V \bigoplus V$ to be the direct sum. Then, $F(V) \in Ob(R - \text{Mod})$ since it is also a vector space.

(a-ii) Let $V, W \in K - \text{Mod}$ and $T \in Hom_{K-\text{Mod}}(V, W)$ be a linear transformation.

Let $V \bigoplus V$ and $W \bigoplus W$ be the corresponding objects after the functor has been applied. Then, $\phi : V \bigoplus V \to W \bigoplus W$ is a $R-$module homomorphism.

Let $v_1, v_2 \in V \Rightarrow (v_1, v_2) \in V \bigoplus V$. Then, define induced map as follows:

$$F(T)(v_1, v_2) = \phi(v_1, v_2) = (T(v_1), T(v_2)) \in W \bigoplus W$$

Check Verify that ϕ is indeed a $R-$module homomorphism.

Let $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$, and let $x = (v_1, v_2), y = (\overline{v_1}, \overline{v_2}) \in V \bigoplus V$. Then, we have:

$$\begin{aligned}
\phi(rx + y) &= \phi(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \end{bmatrix}) \\
&= \phi(\begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}) + \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \end{bmatrix}) \\
&= \begin{bmatrix} T(av_1 + bv_2 + \overline{v_1}) \\ T(cv_1 + dv_2 + \overline{v_2} \end{bmatrix} \\
&= \begin{bmatrix} aT(v_1) + bT(v_2) \\ cT(v_1) + dT(v_2) \end{bmatrix} + \begin{bmatrix} T(\overline{v_1}) \\ T(\overline{v_2}) \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} T(v_1) \\ T(v_2) \end{bmatrix}) + \begin{bmatrix} T(\overline{v_1}) \\ T(\overline{v_2}) \end{bmatrix} \qquad \text{follows from linearity of } T \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \phi(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}) + \phi(\begin{bmatrix} \overline{v_1} \\ \overline{v_2} \end{bmatrix}) \\
&= r\phi(x) + \phi(y)
\end{aligned}$$

(a-iii) Let 1_V be the identity morphism on V (in this case, it is the identity linear transformation).

Let $v_1, v_2 \in V$ and $\phi \in Hom_{R-\text{Mod}}(V \bigoplus V, V \bigoplus V)$. Then,

$$\begin{aligned}
F(1_V)(v_1, v_2) &= \phi(v_1, v_2) \\
&= (1_V(v_1), 1_V(v_2)) \\
&= (v_1, v_2)
\end{aligned}$$

$\therefore F(1_V) = 1_{F(V)}$.

(a-iv) Here is the corresponding commutative diagram:

$$\begin{array}{ccc}
V & \xrightarrow{\quad T \quad} & W \\
\big\downarrow F & & \big\downarrow F \\
F(V) = V \bigoplus V & \xrightarrow{\phi} & F(W) = W \bigoplus W
\end{array}$$

Note, there should also be an induced morphism from $T \dashrightarrow \phi$.

(a-v) Let $T_1 \in Hom_{K-\text{Mod}}(V, W), T_2 \in Hom_{K-\text{Mod}}(W, X)$. Let $\phi_1 \in Hom_{R-\text{Mod}}(V \bigoplus V, W \bigoplus W), \phi_2 \in Hom_{R-\text{Mod}}(W \bigoplus W, X \bigoplus X)$ be the corresponding morphisms. Then,

$$
\begin{aligned}
F(T_2 \circ T_1)(v_1, v_2) &= (T_2(T_1(v_1)), T_2(T_1(v_2))) \\
&= F(T_2) \circ (T_1(v_1), T_1(v_2)) \\
&= F(T_2) \circ F(T_1)(v_1, v_2)
\end{aligned}
$$

Hence, F is a covariant functor.

(b) Recall e in a Ring is idempotent if $e^2 = e$. A primitive idempotent is an idempotent e such that eR is indecomposable, i.e. we cannot have $eR = eR_1 \bigoplus eR_2$ with $eR_1, eR_2 \neq 0$.

Let $R - \text{Mod}$ and $K - \text{Mod}$ be categories and $G : R - \text{Mod} \to K - \text{Mod}$.

Show Show that G is a functor i.e.

(1) $\forall\ M \in Ob(R - \text{Mod}), G(M) \in Ob(K - \text{Mod})$.

(2) $\forall\ \phi \in Hom_{R-\text{Mod}}(M_1, M_2), G(\phi) \in Hom_{K-\text{Mod}}(G(M_1), G(M_2))$ such that $G(1_{M_1}) = 1_{G(M_1)}$.

(3) $G(\phi_2 \circ \phi_1) = G(\phi_2) \circ G(\phi)$ for all ϕ_2, ϕ_1 composable in $R - \text{Mod}$.

(b-i) Let $M \in R - \text{Mod}$. Then, define $G(M) = eM$ where $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$is primitive idempotent. Then, $eM \in K - \text{Mod}$.

(b-ii) Let $M_1, M_2 \in R - \text{Mod}$ and $\phi \in Hom_{R-\text{Mod}}(M_1, M_2)$ be a $R-$module homomorphism.

Note, $G(M_1) = eM_1$ and $G(M_2) = eM_2$ are corresponding objects.

Define $G(\phi) = \overline{\phi}$ where $\overline{\phi}(eM_1) = e\phi(M_1)$.

Check (Note we have checked that eM is a left eRe-module and eRe is isomorphic to the field K.)

Verify that $\overline{\phi}$ is an eRe-module homomorphism.

Let $m_1, m_2 \in M, R_1 \in R$. Then, we have

1.

$$
\begin{aligned}
\overline{\phi}(em_1 + em_2) &= \phi(em_1) + \phi(em_2) \quad && \text{since } \phi \text{ is an R-module homomorphism} \\
&= e\phi(m_1) + e\phi(m_2) \\
&= \overline{m_1} + \overline{m_2}
\end{aligned}
$$

2. (Note an element in eRe looks like eR_1e for $R_1 \in R$). Then, we also have

$$
\begin{aligned}
\overline{\phi}(eR_1e(em)) &= eR_1e(e\phi(m)) \\
&= eR_1e\overline{\phi}(em)
\end{aligned}
$$

Hence, $\overline{\phi}$ is an eRe-module homomorphism.

(b-iii) Let 1_{M_1} be the identity morphism on M_1. Then,

$$\begin{aligned} G(1_{M_1})(eM_1) &= \overline{1_{M_1}}(eM_1) \\ &= e1_{M_1}(M_1) \\ &= eM_1 \end{aligned}$$

$\therefore$, $G(1_{M_1}) = 1_{G(M_1)}$.

(b-iv) Here is the corresponding diagram:

$$\begin{array}{ccc} M_1 & \xrightarrow{\phi} & M_2 \\ \downarrow G & & \downarrow G \\ G(M_1) = eM_1 & \xrightarrow{\overline{\phi}} & G(M_2) = eM_2 \end{array}$$

Note, there should also be an induced morphism $\phi \dashrightarrow \overline{\phi}$.

(b-v) Let $\phi_1 \in Hom_{R-\text{Mod}}(M_1, M_2), \phi_2 \in Hom_{R-\text{Mod}}(M_2, M_3)$. Then,

$$\begin{aligned} G(\phi_2 \circ \phi_1)(eM_1) &= \overline{\phi_2 \circ \phi_1}(eM_1) \\ &= e(\phi_2(\phi_1(M_1)) \\ &= \overline{\phi_2}(e\phi_1(M_1)) \\ &= \overline{\phi_2} \circ \overline{\phi_1}(eM_1) \\ &= G(\phi_2) \circ G(\phi_1)(eM_1) \end{aligned}$$

Hence, G is a covariant functor.

(c) Recall that $F : K - \text{Mod} \to R - \text{Mod}$ and $G : R - \text{Mod} \to K - \text{Mod}$.

Then, $FG : R - \text{Mod} \to R - \text{Mod}$. We need to show that FG is isomorphic to the identity functor on $R - \text{Mod}$.

Consider a natural transformation $\eta : FG \to 1_{R-\text{Mod}}$

From Class One can check that a morphism $\eta : FG \to 1_{R-\text{Mod}}$ is an isomorphism $\iff$

$$\eta_M : FG(M) \to 1_{R-\text{Mod}}(M)$$

is an isomorphism for all all $M \in Ob(R - \text{Mod})$.

Let $M \in R - \text{Mod}$. Then, we have the following:

$$
\begin{aligned}
G(M) &= eM \\
F(G(M)) &= F(eM) \\
&= eM \bigoplus eM \\
&= e(M \bigoplus M)
\end{aligned}
$$

On the other hand, $1_{R-\text{Mod}}(M) = M$.

Show Show that $e(M \bigoplus M) \cong M$. Consider $\phi : e(M \bigoplus M) \to M$ by $\psi(e(m_1, m_2)) = m$, where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Show that ψ is (i) 1-1, (ii) onto, and (iii) preserves homomorphisms.

(i)

$$
\begin{aligned}
\psi(e(m_1, m_2)) &= \psi(e(\overline{m_1}, \overline{m_2}) \\
\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \overline{m_1} \\ \overline{m_2} \end{bmatrix} \\
\Rightarrow m_1 &= \overline{m_1}
\end{aligned}
$$

(Note, we also have $0m_2 = 0\overline{m_2} = 0$). Hence, ψ is 1-1.

(ii) Let c be an arbitrary element of M. Then, we can find elements $c, \overline{c} \in M \bigoplus M$ such that $\psi(e(c, \overline{c})) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ \overline{c} \end{bmatrix} = c$.

Since c was arbitrary, we have shown that ψ is onto.

(iii) Show that ψ is a homomorphism.

1. Let $(x_1, x_2), (y_1, y_2) \in M \bigoplus M$. Then,

$$
\begin{aligned}
\psi(e((x_1, x_2) + (y_1, y_2)) &= \psi(e(x_1 + y_1, x_2 + y_2)) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\
&= x_1 + y_1 \\
&= \psi(e(x_1, x_2)) + \psi(e(y_1, y_2))
\end{aligned}
$$

2. Let $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$\psi(r \cdot e(x_1, x_2)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \psi(e(x_1, x_2))$

Conclude $\therefore$, ψ is a $R - \text{Mod}$ homomorphism and we have shown that $e(M \bigoplus M) \cong M)$.